



**Cambridge Assessment International Education**  
Cambridge International Advanced Level

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**FURTHER MATHEMATICS**

**9231/22**

Paper 2

**October/November 2018**

MARK SCHEME

Maximum Mark: 100

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**Published**

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

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This document consists of **14** printed pages.

**PUBLISHED****Generic Marking Principles**

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

**GENERIC MARKING PRINCIPLE 1:**

Marks must be awarded in line with:

- the specific content of the mark scheme or the generic level descriptors for the question
- the specific skills defined in the mark scheme or in the generic level descriptors for the question
- the standard of response required by a candidate as exemplified by the standardisation scripts.

**GENERIC MARKING PRINCIPLE 2:**

Marks awarded are always **whole marks** (not half marks, or other fractions).

**GENERIC MARKING PRINCIPLE 3:**

Marks must be awarded **positively**:

- marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate
- marks are awarded when candidates clearly demonstrate what they know and can do
- marks are not deducted for errors
- marks are not deducted for omissions
- answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.

**GENERIC MARKING PRINCIPLE 4:**

Rules must be applied consistently e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.

**GENERIC MARKING PRINCIPLE 5:**

Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).

**GENERIC MARKING PRINCIPLE 6:**

Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

**Abbreviations**

AEF/OE Any Equivalent Form (of answer is equally acceptable) / Or Equivalent

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

CAO Correct Answer Only (emphasising that no “follow through” from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous’ answer

ISW Ignore Subsequent Working

SOI Seen or implied

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Question	Answer	Marks	Guidance
1	$v_A^2 = \omega^2 (a^2 - 0.1^2)$ and $v_B^2 = \omega^2 (a^2 - 0.5^2)$	<b>B1</b>	Use $v^2 = \omega^2 (a^2 - x^2)$ at <i>A</i> and <i>B</i> (may be implied)
	$a^2 - 0.1^2 = 2 (a^2 - 0.5^2)$	<b>M1</b>	Find amplitude <i>a</i> from ratio 2 of [ $\frac{1}{2} m$ ] $v_A^2$ to [ $\frac{1}{2} m$ ] $v_B^2$
	$a^2 = 0.5 - 0.01 = 0.49$ , $a = 0.7$ [m]	<b>A1</b>	(taking ratio $\frac{1}{2}$ loses A1)
		<b>3</b>	

Question	Answer	Marks	Guidance
2(i)	$2mv_A + mv_B = 2mu$ (AEF)	<b>M1</b>	Use momentum (allow <i>m</i> omitted)
	$v_B - v_A = \frac{2}{3} u$	<b>M1</b>	Use Newton’s law (M0 if LHS signs inconsistent)
	$v_A = 4 u / 9$ , $v_B = 10 u / 9$	<b>A1, A1</b>	Combine to find speeds of <i>A</i> and <i>B</i> after collision
		<b>4</b>	

Question	Answer	Marks	Guidance
2(ii)	$w_B = [-] \frac{1}{2} v_B [= \pm 5 u / 9]$	<b>M1</b>	Relate speed $w_B$ of $B$ after colln. with wall to $v_B$ (ignore sign)
	<i>EITHER:</i> $(d - x) / v_A = d/v_B + x/w_B$ (AEF)	<b>M1</b>	<i>EITHER:</i> Equate times in terms of dist. $x$ from wall to 3rd colln.
	$(d - x)/4 = d/10 + x/5, x = d/3$	<b>M1A1</b>	Substitute for speeds to solve for $x$
	$t = (d - x) / v_A = (2 d / 3) / (4 u / 9) = 3d / 2u$	<b>A1</b>	and hence find reqd. time $t$
	<i>OR:</i> $x_A = (d/v_B) v_A = (9d/10u) / (4u/9) = 2 d / 5$	<b>(M1)</b>	<i>OR:</i> Find dist. $x_A$ moved by $A$ when $B$ reaches wall
	$t = d/v_B + (d - x_A) / (v_A + w_B)$ $= d / (10 u / 9) + (3 d / 5) / (4 u / 9 + 5 u / 9)$ $= 9d / 10u + 3d / 5u$	<b>M1A1</b>	Find $t$ by adding times to and from wall (or equivalent method)
	$= 3d / 2u$	<b>A1)</b>	
		<b>5</b>	

Question	Answer	Marks	Guidance
3(i)	$\frac{1}{2}mv_B^2 = \frac{1}{2}mu^2 - mga (\sin \alpha - \cos \alpha)$ $[v_B^2 = u^2 - (14/13) ag]$	<b>M1A1</b>	Find speed $v_B$ at $B$ by conservation of energy (A0 if no $m$ ) [ $\sin \alpha = 12/13, \cos \alpha = 5/13$ ]
	$[T_B =] mv_B^2/a - mg \sin \alpha = 0 [v_B^2 = (12/13) ag]$	<b>B1</b>	Equate tension $T_B$ at $B$ to zero by using $F = ma$ radially
	$u^2 = (3 \sin \alpha - 2 \cos \alpha) ag$	<b>M1</b>	Combine to verify $u^2$
	$= (36/13 - 10/13) ag = 2 ag$ AG	<b>A1</b>	
			<b>5</b>

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Question	Answer	Marks	Guidance
3(ii)	<i>EITHER:</i> $\frac{1}{2}mv_C^2 = \frac{1}{2}mu^2 + mga(1 + \cos \alpha)$ or $\frac{1}{2}mv_B^2 + mga(1 + \sin \alpha)$ [ $v_C^2 = 62ag/13$ ]	<b>M1</b>	Find $v_C^2$ at lowest point $C$ by conservation of energy
	$T_{max} = mv_C^2/a + mg$	<b>B1</b>	Find tension $T_{max}$ at lowest point from $F = ma$ radially
	$= 62mg/13 + mg$	<b>M1</b>	Combine to find $T_{max}$
	$= 75mg/13$ or $5.77 mg$ or $57.7 m$	<b>A1</b>	
	<i>OR:</i> $\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + mga(\cos \alpha + \cos \theta)$ [ $V^2 = 2ag(18/13 + \cos \theta)$ ]	<b>(M1)</b>	Find $V^2$ at general point by conservation of energy (where $OP$ is e.g. at $\theta$ to downward vertical)
	$T = mV^2/a + mg \cos \theta = (36/13 + 3 \cos \theta) mg$	<b>B1</b>	Find tension $T$ at general point from $F = ma$ radially
	$T_{max} = (36/13 + 3) mg$	<b>M1</b>	Combine to find $T_{max}$ at lowest point where $\theta = 0$
	$= 75mg/13$ or $5.77 mg$ or $57.7 m$	<b>A1)</b>	
	<b>4</b>		

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Question	Answer	Marks	Guidance
4(i)	$A: R_C \times x - W \cos 45^\circ \times a = 0$ $B: F_A \cos 45^\circ \times 2a - R_A \cos 45^\circ \times 2a - R_C \times (2a - x) + W \cos 45^\circ \times a = 0$ $C: F_A \cos 45^\circ \times x - R_A \cos 45^\circ \times x + W \cos 45^\circ \times (x - a) = 0$ $G: F_A \cos 45^\circ \times a - R_A \cos 45^\circ \times a + R_C \times (x - a) = 0$ $D: R_A \cos 45^\circ \times a - R_C \cos 45^\circ \times x \cos 45^\circ - R_C \cos 45^\circ \times (x - a) \cos 45^\circ = 0$	<b>B1</b>	Take moments for rod about one chosen point $(F_A \times$ may be replaced by $\mu R_A$ and $\cos 45^\circ$ by e.g. $1/\sqrt{2}$ ) (A single resolution along the rod will then suffice since no $R_C$ ) $(G$ is mid-point of $AB$ ) $(D$ is on ground below $G$ )
	Horizontally: $F_A - R_C \cos 45^\circ = 0$	<b>B1</b>	Find two more indep. eqns, e.g. resolution of forces on rod
	Vertically: $R_A + R_C \cos 45^\circ - W = 0$	<b>B1</b>	(a second moment eqn. may be used)
	Along rod: $F_A \cos 45^\circ + R_A \cos 45^\circ - W \cos 45^\circ = 0$	<b>(B1)</b>	
	Perp. to rod: $F_A \cos 45^\circ - R_A \cos 45^\circ - R_C + W \cos 45^\circ = 0$	<b>B1)</b>	
	$[R_A = W / (1 + \mu), F_A = \mu W / (1 + \mu), R_C = \mu W \sqrt{2} / (1 + \mu)]$ $x = a (1 + \mu) / 2\mu$ or $\frac{1}{2} a (1 + 1/\mu)$	<b>M1A1</b>	Combine to find $x$ (using $F_A = \mu R_A$ and $\cos 45^\circ = 1/\sqrt{2}$ )
		<b>5</b>	
4(ii)	$a (1 + \mu) / 2\mu \leq 2a$ so $\mu \geq \frac{1}{3}$	<b>M1A1</b>	Verify $\mu$ using $x \leq 2a$
	AG	<b>2</b>	
4(iii)	$a (1 + \mu) / 2\mu = 3a/2$ so $\mu = \frac{1}{2}$	<b>M1A1</b>	Find $\mu$ when $x = 3a/2$ using result in (i)
	$F_A = W/3, R_A = 2W/3 [R_C = (\sqrt{2}) W/3]$ $N_A = \sqrt{(F_A^2 + R_A^2)} = (\sqrt{5}/3) W$ or $0.745 W$	<b>M1A1</b>	Find $F_A, R_A$ and hence magnitude of resultant force $N_A$ at $A$
		<b>4</b>	





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Question	Answer	Marks	Guidance
6	$\bar{x} = 90.3 / 8 = 11.2875$ (to 4 s.f.)	<b>B1</b>	Find sample mean
	$s^2 = (1043.67 - 90.3^2/8) / 7$	<b>M1</b>	Estimate population variance
	$= 19\,527 / 5600$ or 3.487 [or 1.867 <sup>2</sup> ] (to 4 s.f.)	<b>A1</b>	(allow biased here: 3.051 or 1.747 <sup>2</sup> )
	$90.3 / 8 \pm t \sqrt{(s^2/8)}$	<b>M1</b>	Find confidence interval
	$t_{7, 0.975} = 2.365$ (to 4 s.f.)	<b>A1</b>	State or use correct tabular value of $t$
	$11.3 \pm 1.6$ or [9.7, 12.8[5]]	<b>A1</b>	Evaluate C.I. (either form)
		<b>6</b>	

Question	Answer	Marks	Guidance
7(i)	<i>EITHER:</i> $G(y) [= P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})] = (1/90)(y + y^2)$	<b>M1A1</b>	Find or state $G(y)$ for $0 \leq x \leq 3$ from $Y = X^2$ (allow $<$ or $\leq$ throughout)
	<i>OR:</i> Use $x = y^{1/2}$ to find $f(x) = (1/90)(2x + 4x^3) = (1/90)(2y^{1/2} + 4y^{3/2})$ and $dx/dy = 1 / 2y^{1/2}$	<b>(M1A1)</b>	Find $f(x)$ and $dx/dy$ for use in $g(y) = f(x) \times  dx/dy $
	$g(y) [= G'(y)] = (1/90)(1 + 2y)$	<b>A1</b>	Find $g(y)$ in simplified form
	for $0 \leq y \leq 9$ [ $g(y) = 0$ otherwise]	<b>A1</b>	State corresponding range of $y$ at any stage
		<b>4</b>	
7(ii)	$E(Y) = (1/90) \int (y + 2y^2) dy$	<b>M1</b>	Find mean of $Y$ from $\int y g(y) dy$
	$= (1/90) [\frac{1}{2} y^2 + \frac{2}{3} y^3]_0^9 = 117/20$ or 5.85	<b>A1</b>	
		<b>2</b>	

Question	Answer	Marks	Guidance
8(i)	$P(X \geq 5) = (1 - p)^4 = 0.4096$ $p = 1 - 0.8 = 0.2$ AG	<b>M1A1</b>	Verify $p$ using $P(X \geq 5) = 0.4096$
		<b>2</b>	
8(ii)	$P(X = 6) = (1 - p)^5 p = 0.8^5 \times 0.2 = 0.0655$	<b>M1A1</b>	Find $P(X = 6)$
		<b>2</b>	
8(iii)	$1 - (1 - p)^N > 0.9$	<b>M1</b>	Formulate condition for $N$ ( $1 - (1 - p)^{N-1}$ is M0)
	$0.1 > 0.8^N$	<b>A1</b>	(< or = can earn M1 M1 only, max 2/4)
	$N > \log 0.1 / \log 0.8 = 10.3$	<b>M1</b>	Rearrange and take logs (any base) to give bound
	$N_{\min} = 11$ corresponding to Monday of 3rd week	<b>A1</b>	Find $N_{\min}$ and corresponding day and week
		<b>4</b>	

Question	Answer	Marks	Guidance
9(i)	$H_0: \rho = 0, H_1: \rho \neq 0$	<b>B1</b>	State both hypotheses (B0 for $r \dots$ )
	<i>EITHER:</i> $r_{5,5\%} = 0.878$	<b>*B1</b>	State or use correct tabular two-tail $r$ -value
	Accept $H_0$ if $ r  < \text{tab. value}$ (AEF)	<b>M1</b>	State or imply valid method for conclusion
	<i>OR:</i> $t_r = r\sqrt{(n-2)/(1-r^2)} = -2.08, t_{4,0.975} = 2.776 \text{ or } 2.78$	<b>(*B1)</b>	(Rarely seen)
	Accept $H_0$ if $ t_r  < \text{tab. } t\text{-value}$ (AEF)	<b>M1</b>	
	No evidence of [non-zero] correlation (AEF)	<b>A1</b>	Correct conclusion (dep *B1)
		<b>4</b>	



Question	Answer	Marks	Guidance
10(ii)	$E_4 = 40 \lambda^4 e^{-\lambda} / 4!$ with $\lambda = 2.95$ [= $40 \times 0.1652$ ] (allow $(\lambda/4) \times E_3 = 0.7375 \times 8.96$ )	<b>M1A1</b>	State expression for reqd. expected value $E_4$
		<b>2</b>	
10(iii)	$H_0$ : [Poisson] distribution fits data (AEF)	<b>B1</b>	State (at least) null hypothesis in full
	$O_i$ :    8        8        10        5        9 $E_i$ : <u>8.27</u> 9.11    8.96    6.61 <u>7.05</u>	<b>M1</b>	Combine values consistent with all exp. values $\geq 5$
	$X^2 = 0.0088 + 0.1352 + 0.1207 + 0.3921 + 0.5394$	<b>M1</b>	Find value of $X^2$ from $\Sigma (E_i - O_i)^2 / E_i$ [or $\Sigma O_i^2 / E_i - n$ ]
	= 1.20 (to 3 s.f.)	<b>A1</b>	
	No. $n$ of cells:    8        7        6        5 $\chi_{n-2, 0.95}^2$ :        12.59    11.07    9.488 <u>7.815</u>	<b>B1</b>	State or use consistent tabular value $\chi_{n-2, 0.95}^2$ (to 3 s.f.) [ <b>FT</b> on number, $n$ , of cells used to find $X^2$ ]
	Accept $H_0$ if $X^2 <$ tabular value (AEF)	<b>M1</b>	State or imply valid method for conclusion
	1.20 [ $\pm 0.1$ ] $<$ 7.81[5] so [Poisson] distn. fits [data] or distn. is a suitable model (AEF)	<b>A1</b>	Conclusion (requires both values correct)
	<b>7</b>		

Question	Answer	Marks	Guidance
11A(i)	<i>EITHER:</i> $40 e_0 / 0.8 = 2g$ , $e_0 = 0.4$ [or $OP_0 = 1.2$ ] [m]	<b>M1A1</b>	Find extension $e_0$ [or $OP_0$ ] at equilibrium position $P_0$
	$2 \frac{d^2x}{dt^2} = -40(e_0 + x) / 0.8 + 2g$ or $= +40(e_0 - x) / 0.8 - 2g$	<b>M1A1</b>	Use Newton's law at general point (e.g. $x$ below or above $P_0$ ) (ignore LHS sign here only)
	<i>OR:</i> $2 \frac{d^2y}{dt^2} = -40(y - 0.8) / 0.8 + 2g$ $= -50y + 60$	<b>(M1A1)</b>	Use Newton's law at general point in terms of $y = OP$ (ignore LHS sign here only)
	Take $x = y - 1.2$ to give	<b>M1A1)</b>	Change variable to give standard form of SHM eqn
	$\frac{d^2x}{dt^2} = -25x$	<b>A1</b>	Hence SHM (A0 if wrong sign or LHS unclear) (B1 only for stating SHM eqn. without proof)
	$T = 2\pi / \sqrt{(25)} = 2\pi/5$ <span style="float: right;">AG</span>	<b>A1</b>	Verify period $T$ using $T = 2\pi/\omega$ with $\omega = \sqrt{(25)}$
	$OP_0 = 1.2$ [m]	<b>B1</b>	State $OP_0$ explicitly (may imply first M1 A1)
		<b>7</b>	
11A(ii)	$0.4^2 = 25(a^2 - 0.06^2)$	<b>M1A1</b>	Find amplitude $a$ from $v^2 = \omega^2(a^2 - x^2)$
	$a = \sqrt{(0.0064 + 0.0036)} = 0.1$ [m]	<b>A1</b>	
		<b>3</b>	
11A(iii)	$40 e_1 / 0.8 = (2 + M) g$ , <span style="float: right;"><math>[e_1 = 0.4 + 0.2 M]</math></span> or $40(e_1 - e_0) / 0.8 = Mg$ <span style="float: right;"><math>[e_1 - e_0 = 0.2 M]</math></span>	<b>M1A1</b>	Find extension $e_1$ [or $OP_1$ ] at first equilibrium position $P_1$ or equate additional extension to $M$ by Hooke's Law
	$e_1 - e_0 = a$ , $M = a/0.2 = 0.5$	<b>M1A1</b>	Find $M$ by relating $e_0$ , $e_1$ and $a$
		<b>4</b>	

Question	Answer	Marks	Guidance
11B(i)	$H_0: \mu_x = \mu_y$ , $H_1: \mu_x > \mu_y$ (or in terms of $\mu_A$ , $\mu_B$ )	<b>B1</b>	State hypotheses (B0 for $\bar{x}$ ...)
	$s_x^2 = (14.1775 - 10.56^2/8) / 7 = 0.03404$ and $s_y^2 = (15.894 - 12.39^2/10) / 9 = 0.06031$ (to 3 s.f.)	<b>M1A1</b>	Estimate both population variances (allow biased here: 0.02979 and 0.05428)
	$s^2 = (7 s_x^2 + 9 s_y^2) / 16$ or $(14.1775 - 10.56^2/8 + 15.894 - 12.39^2/10) / 16$ $= 0.78109 / 16$ or 0.04882 or 0.22095 <sup>2</sup> (to 3 s.f.)	<b>M1A1</b>	Find pooled estimate of common variance (M1 A1 for $s_x^2$ and $s_y^2$ may be implied here)
	$t_{16, 0.9} = 1.337$ (to 3 s.f.)	<b>*B1</b>	State or use correct tabular $t$ value
	$t = (1.32 - 1.239) / s\sqrt{(1/10 + 1/8)} = 0.773$	<b>M1A1</b>	Calculate value of $t$ (or $-t$ ) (or can compare $\bar{y} - \bar{x} = 0.081$ with 0.140)
	$t <$ tabular value so claim not justified or $A$ 's not heavier than $B$ 's (AEF)	<b>B1</b>	Correct conclusion ( <b>FT</b> on $t$ , dep *B1)
	<b>SC:</b> $z = (1.32 - 1.239) / \sqrt{(s_x^2/8 + s_y^2/10)}$ $= 0.081 / \sqrt{(0.078)} = 0.799$	<b>(B1)</b>	<b>SC:</b> Implicitly taking $s_x^2$ , $s_y^2$ as unequal popln. variances (may also earn first B1 M1 A1)
	$z < 1.282$ so claim is not justified (AEF)	<b>B1)</b>	Comparison with $z_{0.9}$ and conclusion ( <b>FT</b> on $z$ ; max 5/9)
		<b>9</b>	
11B(ii)	$\bar{x} = 1.28$ and $s^2 = 0.294 / 7 [= 0.042$ (or $0.205^2$ )]	<b>M1</b>	Find sample mean & estimate popn. var (allow M1 if $0.294 / 8$ )
	$t = (1.28 - p) / \sqrt{(0.042/8)}$ (AEF)	<b>M1A1</b>	Find value of $t$
	$t > 1.415$ (< is A0), $p < 1.28 - 0.1025$ , $p_{\max} = 1.18$	<b>M1A1</b>	Find $p_{\max}$ by comparison with tabular value, here $t_{7, 0.9}$
		<b>5</b>	